701. One of the following statements is true; the other is not. Identify and disprove the false statement.

(a) $(x-k)\cos x = 0 \implies x = k$,
(b) $(x-k)2^x = 0 \implies x = k$.

702. The variable X has a normal distribution. The distribution of the means of samples of size 20 is $\bar{X} \sim N(50, 1.25)$. Give the mean and variance of the population.

703. A function is defined, for $x \neq 0$, by

$$\mathrm{f} : x \mapsto \frac{4}{x^2} + x^3 + 4x + 7.$$

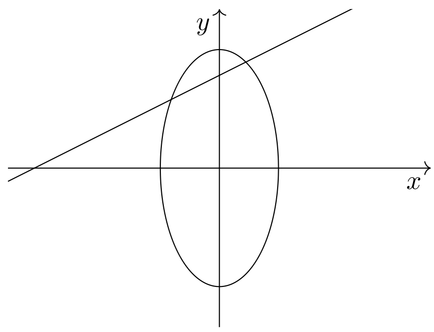
(a) Show that $\mathrm{f}'(2) = 15$ and evaluate $\mathrm{f}(2)$.
(b) Hence, show that the linear function that best approximates $\mathrm{f}(x)$ at $x = 2$ is $\mathrm{g}(x) = 15x - 6$.

704. Solve $\cos^2 x - \cos x = 0$, for $x \in [-180°, 180°]$.

705. A student writes: "If the vertex of a positive parabola is at (a, b), then the parabola must have equation $y = (x-a)^2 + b$." Correct the error.

706. A particle moves with position given by $r = t^3 - t^2$, for the time period $t \in [0, 5]$. Show that, at time $t = \frac{1}{3}\left(1+\sqrt{61}\right)$, its instantaneous speed is the same as its average speed over the five seconds.

707. The diagram shows an ellipse and a straight line, with equations $4x^2 + y^2 = 5$ and $2x - 4y + 7 = 0$:



Determine the coordinates of the intersections, giving your answers in exact form.

708. Disprove the following: "No term in the sequence $u_n = n^2 + 2$ is a perfect cube."

709. Two functions f and g are such that the indefinite integral of their difference is linear. Prove that, if $\mathrm{f}(x) = \mathrm{g}(x)$ for any x, then $\mathrm{f}(x) = \mathrm{g}(x)$ for all x.

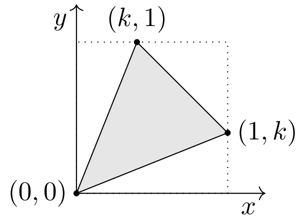
710. The interior angles of an irregular pentagon are in AP. The smallest is $\frac{\pi}{4}$ radians. Find the largest.

711. The following is an equation in the variable a:

$$\left[a^x - a^{x-1}\right]_{x=0}^{x=1} = 0.$$

(a) Show that $a - 2 + a^{-1} = 0$.
(b) Solve to find a.

712. An isosceles triangle is set up, inside a unit square, with one vertex at the origin.



(a) Show that the area of the shaded triangle is given by $A = \frac{1}{2}(1-k^2)$.
(b) Find the rate of change of A with respect to k.
(c) Hence, or otherwise, show that the maximal area of such a triangle is $\frac{1}{2}$.

713. Show that the functions $\mathrm{f}(x) = 4x^2 + 8x + 2$ and $\mathrm{g}(x) = x^2 + 6x + 7$ have the same range over $\mathbb{R}$.

714. Three inequalities are given as

$$\begin{aligned} 3x - 5y &\geq 10, \\ x + 2y &\leq 15, \\ x + 5y &\geq 24. \end{aligned}$$

By solving pairs of equations, or otherwise, show that there is only one integer solution (x, y, z) that satisfies all three inequalities.

715. The iteration $u_{n+1} = u_n + n$, $u_1 = 1$, defines a quadratic sequence.

(a) Find the first five terms.
(b) The ordinal definition is $u_n = an(n-1) + b$. Find the constants a and b.

716. Either prove or disprove the following statement:

$$\int_a^b \mathrm{f}(x)\,dx + \int_b^c \mathrm{f}(x)\,dx = \int_a^c \mathrm{f}(x)\,dx.$$

717. By eliminating the variable t from the formulae $s = ut + \frac{1}{2}at^2$ and $v = u + at$, derive $v^2 = u^2 + 2as$.

718. Write each of the following as a single interval:

(a) $(-\infty, 2] \cup (-4, 6]$,
(b) $[0, \infty) \setminus (1, \infty)$,
(c) $(-\infty, 1] \cap [-1, \infty)$.

719. The line $x = k$ is normal to $y = x^4 - 32x^2$. Find all possible values of the constant k.

720. Show that $0.\dot{2}3\dot{7}$ may be written as $\dfrac{237}{999}$.

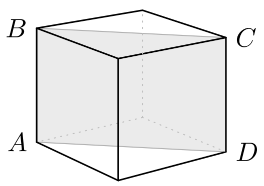
721. The *work done* on an object, moving from $x = a$ to $x = b$ under the action of a force F, is given by

$$W = \int_a^b F\,dx.$$

An object falls from a small height h to ground level. Show that the work done by gravity is mgh.

722. A polynomial function f has $\mathrm{f}(2) = 3$, $\mathrm{f}'(2) = 0$ and $\mathrm{f}''(2) = 3$. Sketch the graph $y = \mathrm{f}(x)$ for x values close to 2.

723. The diagram shows a cube.



Show that rectangle $ABCD$ has the dimensions of a sheet of A4 paper. (A4 paper, when folded in half, produces two sheets of A5 paper, each of which is similar to the original sheet.)

724. State, with a reason, whether these hold:

(a) $x, y \in \mathbb{Q} \implies xy \in \mathbb{Q}$,

(b) $x, y \in \mathbb{R} \setminus \mathbb{Q} \implies xy \in \mathbb{R} \setminus \mathbb{Q}$.

725. To find the gradient formula $\frac{dy}{dx}$ of $y = x^n$, where $n \in \mathbb{N}$, we set up the limit

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}.$$

(a) Using the binomial expansion, show that

$$(x+h)^n \equiv x^n + nx^{n-1}h \\ + \text{ terms in } h^2 \text{ and higher}$$

(b) Hence, prove that $\dfrac{dy}{dx} = nx^{n-1}$.

726. Two dice are rolled. State which, if either, of the following events has the greater probability:

① the total is six,

② the total is eight.

727. A particular cubic function $\mathrm{h}(x) = x^3 + ax^2 + bx + c$ has $\mathrm{h}(0) = 0$, $\mathrm{h}'(0) = 0$, $\mathrm{h}''(0) = 2$.

(a) Find a, b, c,

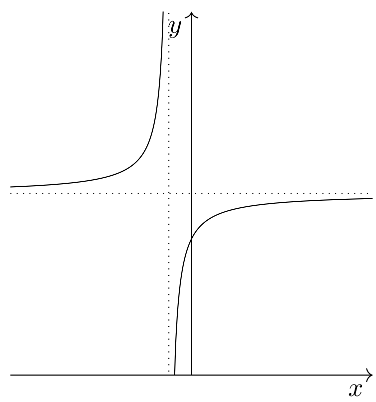
(b) Sketch the graph $y = \mathrm{h}(x)$.

728. Two stationary cows are standing on flat ground, pushing against each other. The contact forces between the cows are horizontal. With reference to Newton's laws, explain how you know that the magnitudes of the following are equal:

(a) the frictional force of the ground on cow A; the force of cow B on cow A,

(b) the force of cow A on cow B; the force of cow B on cow A.

729. The graph below shows the curve $y = \dfrac{8x+3}{2x+1}$:



Find the equations of the two asymptotes.

730. Complete the square in $\sqrt{2}x^2 + \sqrt{8}x + \sqrt{32}$.

731. A sample $\{x_i\}$ of size 12 has mean 10 and variance 25. Write down the mean value of

(a) $x_i - 10$,

(b) $(x_i - 10)^2$.

732. Simplify $\dfrac{d}{dx}(2x + 3y + 5)$.

733. Write down the largest real domain over which each of the following functions may be defined:

(a) $x \mapsto x^1$,

(b) $x \mapsto x^0$,

(c) $x \mapsto x^{-1}$.

734. Give the acceleration of a lift if accurate weighing scales placed inside it overestimate mass by 20%.

735. True or false?

(a) $\displaystyle\int_a^b k\,\mathrm{f}(x)\,dx \equiv k \int_a^b \mathrm{f}(x)\,dx$,

(b) $\displaystyle\int_a^b k + \mathrm{f}(x)\,dx \equiv k + \int_a^b \mathrm{f}(x)\,dx$.

736. A curve has equation

$$x^2 + 4x + y^2 - 2y = 0.$$

A normal is drawn to the curve at $(-4, 0)$. Show, without using calculus, that the equation of the normal is $2y = x + 4$.

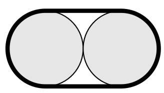
737. A sequence is given iteratively by

$$u_{n+1} = 4u_n, \quad u_1 = 5.$$

Find the value of the first term over one billion.

738. Simplify $\{x : 0 \leq x \leq 3\} \cup \{x : |x - 3| < 2\}$.

739. Two bamboo canes, each of radius 0.5 cm, are bound together, in equilibrium, with a rubber band, as shown below in plan view. The tension in the band is modelled as a constant 3 N throughout. In the plane shown, no external forces act.



(a) Determine the length, in cm, of the band.

(b) Explain how you know that

i. even if the canes are rough, there can be no friction acting between them,

ii. the contact between canes and rubber band has been modelled as smooth.

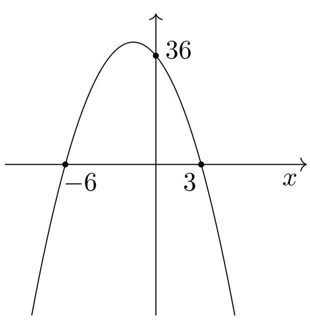
(c) Find the contact force between the two canes.

740. Prove the sine rule.

741. Find the length of the line segment

$$x = 2 + s, \quad y = -1 + \tfrac{4}{3}s, \quad s \in [-1, 2].$$

742. A negative parabola passes through points $(-6, 0)$, $(0, 36)$, and $(3, 0)$.



Find the equation of the graph, giving your answer in the form $y = ax^2 + bx + c$.

743. Four cards are dealt, with replacement, from a standard deck. Find the probability that the first two are hearts, and the second two are diamonds.

744. Find simplified expressions for the sets

(a) $\{x \in \mathbb{R} : |x| < 1\} \cap \{x \in \mathbb{R} : |x| < 2\}$,

(b) $\{x \in \mathbb{R} : |x| < 1\} \cap \{y \in \mathbb{R} : |y| < 2\}$.

745. Show that $\cos^2 x + \cos x - 6 = 0$ has no roots.

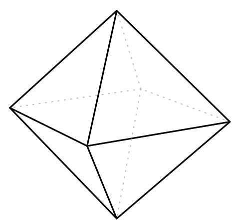
746. A string has tensions applied to each end of it, with magnitudes T_1 and T_2. Explain, with reference to Newton's laws, why, if the string is modelled as light, then $T_1 = T_2$.

747. "The curves $x^2 + y^2 = 1$ and $(x+1)^2 + (y+1)^2 = 1$ are tangent to one another." True or false?

748. Explain why one of the following expressions is well-defined and the other is not:

$$\left.\frac{x^2 - 1}{x^2 - x}\right|_{x=1} \qquad \lim_{x \to 1} \frac{x^2 - 1}{x^2 - x}$$

749. A regular octahedron is shown below.



State, with justification, the type of polyhedron which has vertices at the midpoints of each of the octahedron's faces. (This is called the *dual*.)

750. Using the quadratic discriminant Δ, or otherwise, show that the line $y = 2px - p^2$, for constant p, is tangent to the curve $y = x^2$.

751. Show that a nanocentury is around π seconds.

752. In a probability model, p and $q = 1 - p$ satisfy the equation $16p^2q^2 = 1$.

(a) Show that this equation can be factorised as a difference of two squares.

(b) Solve to find p and q.

753. Euler's number is a constant: $e \approx 2.7$. Write down the range of h, where $\text{h}(x) = (e^x - 1)^2 + 4$.

754. The vector equation of a line L is given, relative to an origin, by $\mathbf{r} = t\mathbf{a} + (1 - t)\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are the position vectors of points A and B respectively, and $t \in \mathbb{R}$ is a parameter.

(a) Show, by giving values of the parameter, that L passes through both A and B.

(b) Find the position vector of a point dividing AB in the ratio $1 : 3$.

755. Prove that no function has linear first and second derivatives that are non-constant.

756. An equation is defined, for constants $a, b \neq 0$, as

$$(ax + b)(bx + a) = 0.$$

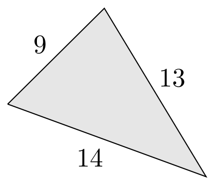
You are given that this equation has precisely one real root. Show that $a^2 = b^2$.

757. Simplify the following:

(a) $\dfrac{p-q}{q-p}$,

(b) $\dfrac{q^2-p^2}{p-q}$.

758. The triangle shown has sides $(9, 13, 14)$ cm.



Show that the triangle has area $18\sqrt{10}$ cm^2.

759. Prove Pythagoras's theorem in three dimensions from Pythagoras's theorem in two dimensions.

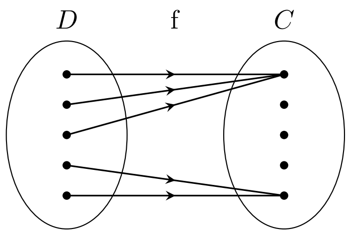
760. Without a calculator, evaluate $\displaystyle\sum_{i=1}^{3} \cos \frac{\pi \text{ rad}}{i}$.

761. A pirate stands in a lift, with a large red parrot on his shoulder. The pirate has mass 80 kg, the parrot has mass 5 kg, and the lift has upwards acceleration a ms^{-2}.

(a) In the case where $a = 0$, determine the force exerted by the pirate's feet on the lift floor.

(b) In the case where $a = \frac{1}{5}g$, determine the force exerted by the parrot's feet on the pirate's shoulder.

762. A function f maps a domain D to a codomain C, according to the following scheme:



You are given that n is the number of roots of the equation $\mathrm{f}(x) = a$, where $a \in C$. Write down all possible values of n.

763. Solve the equation $\sin^2 x + \sin x = 0$, giving all values $x \in [0, 360°)$.

764. Graphs G_1 and G_2 have equations $y = x^2 + kx + k$ and $y = x + k$. Show that, for all $k \in \mathbb{R}$, G_1 and G_2 always have at least one point of intersection.

765. A sample $\{x_i\}$ of size 100 is taken. The sample having been taken, twenty of the data are selected at random. The values of these twenty data are then reduced, each by twenty-five percent. Find the expected percentage change in $\bar{x}$.

766. A quadrilateral has the following properties:

- two of the interior angles are right-angles,
- none of the sides are parallel.

Prove that there is a circle which passes through all four vertices.

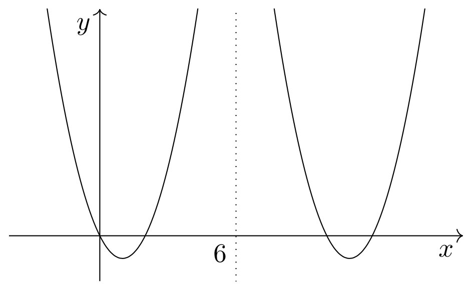
767. Prove that, if α is a fixed point of an invertible function f, then it is also a fixed point of f^{-1}.

768. A heavy box of mass 60 kg is sitting on the ground. The coefficient of friction between it and the ground is $\mu = 0.4$. Find the acceleration of the box if a horizontal force is applied of magnitude

(a) 300 N,

(b) 200 N.

769. In the diagram, the parabola $y = x^2 - 2x$ has been reflected in the line $x = 6$:



Find the equation of the new parabola.

770. Three six-sided dice are stacked neatly on top of one another on a table. Show that the maximum possible number of dots visible is 48.

771. In a trial, it is given that two events A and B have probabilities satisfying

$$\begin{aligned} \mathbb{P}(A \cap B') &= x, \\ \mathbb{P}(B) &= 3x + \tfrac{1}{5}, \\ \mathbb{P}(A' \cap B') &= 4x. \end{aligned}$$

(a) Find x.

(b) Assuming independence, find $\mathbb{P}(A')$.

772. Give the meaning of the following adjectives used in mechanical modelling:

(a) "inextensible",
(b) "uniform",
(c) "rigid".

773. Write 2^t in terms of 16^t.

774. Simultaneous equations A, B, C are as follows:

$$\begin{aligned} A: &\quad x - 4y + 3z + 4 = 0, \\ B: &\quad 3x + y - 4z + 25 = 0, \\ C: &\quad 6x + 3y - 2z + 9 = 0. \end{aligned}$$

(a) Eliminate x from equation-pairs (A, B) and (B, C) to give two equations in y and z.
(b) Solve to find x, y, z.

775. Prove that $\sin 60° = \frac{\sqrt{3}}{2}$.

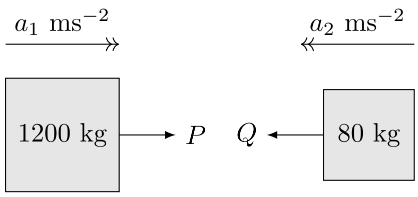
776. You are told that there are infinitely many (x, y) pairs which satisfy both of the following equations

$$2x - 3y = 7, \quad x = \tfrac{3}{2}y + k.$$

Find the value of k.

777. A family of lines L_n is given by $y = mx + c_n$, where c_n forms an arithmetic sequence. Prove that the distance between adjacent lines is constant.

778. A spacecraft is winching an astronaut in using a light, inextensible cable. Initially, the astronaut is floating a constant 50 metres from the spacecraft.



(a) State, with a reason in each case, whether the following pairs of magnitudes are equal:
  i. P and Q,
  ii. a_1 and a_2.
(b) The winch pulls for five seconds, exerting a constant tension of 60 N, and then cuts out. Find the total time taken for the astronaut to reach the spacecraft.

779. Find $\text{f}(0)$, if f is a linear function such that

$$\int_{-2}^{2} \text{f}(x)\, dx = 1.$$

780. In each case, is the following statement true or false? "Negating every data value negates...

(a) ...the mean."
(b) ...the IQR."
(c) ...the median."
(d) ...the standard deviation."
(e) ...the variance."

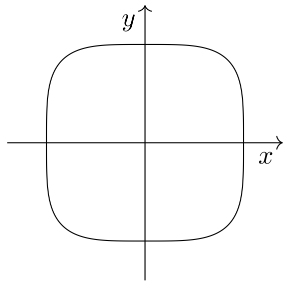
781. You are told that the variables x, y, z satisfy

$$\frac{dy}{dz} \propto \frac{dx}{dz}.$$

State, with a reason, whether each of the following statements is necessarily true:

(a) $y \propto x$,
(b) y and x are linearly related.

782. The graph shows the curve $x^4 + y^4 = 1$:



Determine whether the point $(0.8, 0.9)$ lies inside, on, or outside this curve.

783. A student is asked to determine the solution of the equation $2^{3x} = 5 \cdot 2^{2x}$, showing detailed reasoning. Her solution is as follows:

"From $2^{3x} = 5 \cdot 2^{2x}$, we divide both sides by 2^{2x}, which gives $2^x = 5$. Hence, $x = \log_2 5$."

Criticise the argument.

784. Give the acceleration of a lift if accurate weighing scales placed inside it overestimate mass by $k\%$.

785. The following line segments, when drawn together in the (x, y) plane, depict a capital letter:

$$\begin{aligned} \mathbf{r} &= s\mathbf{i} + (8 - s)\mathbf{j}, \quad s \in [0, 8], \\ \mathbf{r} &= t\mathbf{i} + t\mathbf{j}, \quad t \in [-4, 4]. \end{aligned}$$

Sketch the line segments and identify the letter.

786. For small values of θ in radians, the cosine function may be approximated by $\cos\theta \approx 1 - \frac{1}{2}\theta^2$. Find the percentage error when this approximation is used to estimate

$$\int_0^{\frac{\pi}{6}} \cos\theta\, d\theta.$$

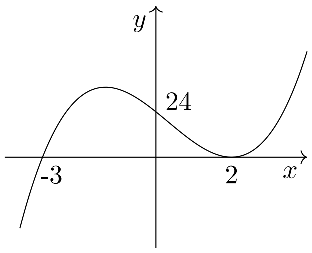
787. Solve $(x + 1) + \dfrac{(x + 1)^2}{x^2 + 1} = 0$.

788. State which, if any, of the implications $\Longrightarrow$, $\Longleftarrow$ or $\Longleftrightarrow$ links the following statements concerning a real number x:

① $x \in [0, 2]$,
② $x \in [1, 3]$.

789. The linear expression $(2x - \beta)$ divides exactly into the quadratic expression $2x^2 + x + \beta$. Find all possible values of β.

790. Find the equation of the cubic curve shown below, on which the axes intercepts have been marked, giving your answer in expanded polynomial form.



791. You are given that non-zero vectors $\mathbf{p}$ and $\mathbf{q}$ are non-parallel. Prove that there are no non-zero real numbers a and b such that $a\mathbf{p} + b\mathbf{q} = 0$.

792. By integrating twice, find the general solution of the following differential equation, for constant a:

$$\frac{d^2y}{dx^2} = 2a.$$

793. You are given the following constant acceleration formulae:

① $v^2 = u^2 + 2as$,
② $s = \frac{1}{2}(u + v)t$.

From these, derive the equation $s = ut + \frac{1}{2}at^2$.

794. The following identity is proposed:

$$(x+1)^4 \equiv \left(A(x-1)^2 + Bx^2\right)^2.$$

Prove that there are no constants A, B which make the above identity hold.

795. The scores A and B on two tests, out of 40 and 60 marks respectively, are to be combined into a single mark X, given out of a hundred. Each test (as opposed to each mark) is to carry equal weight. Find a formula for X in terms of A and B.

796. A positive geometric progression with nth term u_n begins $a, ar, ar^2, \ldots$.

(a) Show that each of the following expressions is non-negative:

i. $1 - 2r + r^2$,

ii. $a - 2ar + ar^2$.

(b) Hence, prove that u_2 cannot be bigger than the mean of u_1 and u_3.

797. State, with a reason in each case, whether the line $x = p$ intersects the following curves:

(a) $y = \dfrac{1}{x+p}$,

(b) $y = \dfrac{1}{x-p}$.

798. Prove that the sum of three odd squares is odd.

799. State, with a reason, whether getting two cards of different suits is more probable if the cards are picked

① with replacement,
② without replacement.

800. Functions f and g are such that $x = a$ is a root of $\mathrm{f}(x) = 0$, $x = b$ is a root of $\mathrm{g}(x) = 0$, and $x = c$ is a root of $\mathrm{f}(x) = \mathrm{g}(x)$. State, with a reason, whether the following implications hold:

(a) If $a = b$, then $\mathrm{f}(c) = 0$.

(b) If $a = c$, then $\mathrm{g}(c) = 0$.

——— End of 8th Hundred ———